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Nagata's research in dimension theory

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ABSTRACT

We give a survey of Jun-iti Nagata's many contributions to dimension theory: characterizations of n-dimensional metric spaces in terms of a special base; characterizations of n-dimensional metrizable spaces in terms of a special metric; imbedding theorems and universal spaces for n-dimensional metric spaces; countable-dimensional metric spaces; dimension and rings of continuous functions; dimension theory beyond metric spaces.

0. Introduction

Dimension theory has been described, by A.H. Stone I believe, as one of the great success stories of general topology. There are three classical definitions of dimension, no one of which is obvious; these are: **ind** (*small inductive dimension*, due independently to Menger and Urysohn), **Ind** (*large inductive dimension*, due to Brouwer and Čech), and **dim** (*covering dimension*, due to Čech and Lebesgue). Precise definitions are as follows:

- $\operatorname{ind} X = -1$ if and only if $X = \emptyset$; $\operatorname{ind} X \leqslant n$ if for every $p \in X$ and every open neighborhood U of p, there is an open neighborhood V of p such that $p \in V \subseteq U$ and $\operatorname{ind}(\overline{V} - V) \leqslant n-1$; $\operatorname{ind} X = \infty$ if $\operatorname{ind} X \leqslant n$ fails for all $n \geqslant -1$;
- $\operatorname{Ind} X = -1$ if and only if $X = \emptyset$; $\operatorname{Ind} X \leqslant n$ if for every closed set H in X and every open set U with $H \subseteq U$, there is an open set V such that $H \subseteq V \subseteq X$ and $\operatorname{Ind}(\overline{V} - V) \leqslant n-1$;
- $\dim X = -1$ if and only if $X = \emptyset$; $\dim X \leqslant n$ if every finite open cover $\mathcal{U}$ of X has a finite open refinement $\mathcal{V}$ of order $\leqslant n+1$ (a cover $\mathcal{V}$ of X has *order* n if n is the largest integer such that some point of X is in the intersection of n elements of $\mathcal{V}$; $\mathcal{V}$ *refines* $\mathcal{U}$ if every element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$).

The history of the discovery of suitable definitions of dimension is quite interesting and is described in considerable detail in [45,36,50,49]. A brief outline is as follows. First of all, according to [36], Cantor is the father of dimension theory; his construction in 1877 of a one-to-one mapping from an interval onto a square demolished the prevailing but vague idea that the dimension of a set is somehow related to the least number of parameters required to describe the set. Peano's construction in 1890 of a continuous mapping from an interval onto a square showed that continuity alone is insufficient to capture dimension. A degree of clarification came in 1911, when Brouwer proved that if f is a continuous and one-to-one mapping from $\mathbb{R}^m$ onto $\mathbb{R}^n$, then $m = n$. However, he did not explicitly isolate a topological property satisfied by one of these spaces but not the other when $m \neq n$. Two years later, in 1913, Brouwer introduced an inductive definition of dimension called "Dimensionsgrad" that is a precursor of Ind and is compatible with both ind and Ind for compact metric spaces. Brouwer's "Dimensionsgrad" gives a clear topological invariant that distinguishes $\mathbb{R}^m$ and $\mathbb{R}^n$ when $m \neq n$. For a detailed discussion of the precise relationship between Brouwer's inductive definition of dimension and Ind, see [50, p. 165]. This definition was influenced by earlier remarks by Poincaré, who pointed out the inductive nature of dimension in the following way: a given continuum is three-dimensional if it can be cut by one or more continua of dimension two;

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a given continuum is two-dimensional if it can be cut by one or more continua of dimension one, and so on. However, he was unclear about the notion of a continuum and a cut; indeed, topological spaces, the appropriate setting for the required precise definitions, were not yet available.

In 1911 Lebesgue formulated a precursor to the covering dimension called the *tiling principle*, which can be stated as follows: the n-cube $[0,1]^n$ has the property that whenever $0 < \epsilon < 1$, there is a finite closed cover $\mathcal{F}$ of $[0,1]^n$ by sets of diameter $< \epsilon$ such that every point is in at most $n+1$ elements of $\mathcal{F}$; moreover, there is no such cover $\mathcal{F}$ such that every point is in at most n elements of $\mathcal{F}$. The tiling principle was proved by Brouwer in 1913 and by Lebesgue in 1921, thereby giving an intuitively appealing property that distinguishes $[0,1]^m$ from $[0,1]^n$ for $m \neq n$.

In 1922–1923, Menger and Urysohn independently gave a definition of dimension within the context of metric spaces that is both inductive and local. It should be noted that much earlier Bolzano (1843–1844) had essentially captured the idea of this dimension function, but his ideas were not published until 1948 and so did not influence the search for a suitable definition; see [49] for a detailed discussion Bolzano's ideas. In the 1930's, Čech began the extension of dimension theory beyond separable metric spaces; in particular, he developed theories of dimension for perfectly normal spaces using Ind and for normal spaces using dim. In the case of dim, he recast the tiling principle in terms of open sets and refinements (see [35]).

Dimension theory was first established for compact metric spaces, mainly by Menger [52] and Urysohn, and then for separable metric spaces, mainly by Hurewitz and Tumarkin. This theory is beautifully covered by Hurewitz and Wallman in their 1941 book *Dimension Theory*. Key results are: $\operatorname{ind}\mathbb{R}^n = n$ and $\operatorname{ind} X = \operatorname{Ind} X = \dim X$ for any separable metric space X. The coincidence of these three functions is quite convenient: generally speaking, if one wants to prove that X has dimension $\leqslant n$, use ind; on the other hand, if one wants to prove that an n-dimensional space has some property, you may assume $\operatorname{Ind} X = n$ or $\dim X = n$. At the end of Chapter 1, Hurewitz and Wallman state "there arise grave difficulties in extending dimension theory to more general spaces." (This statement does not seem to take into account the progress of Čech in the 1930's [34,35].)

Fortunately, these grave difficulties can be overcome with an important new idea: paracompactness. The big breakthrough was the 1948 result by A.H. Stone that every metric space is paracompact. The importance of Stone's Theorem for general topology and dimension theory was emphasized by Nagata in his 1971 survey paper [23], where he states that "after the brilliant era of great pioneers like Tychonoff, Urysohn, Alexandroff, Kuratowski, Čech, etc. there was a period of doldrums in general topology until Stone's famous paper ...." Shortly after Stone's Theorem, Nagata, Smirnov, and Bing independently solved the metrization problem by proving that a regular space X is metrizable if and only if it has a σ-locally finite base (σ-discrete for Bing). With these results in place, Katětov [47] and Morita [54] independently extended the major results of dimension theory to the class of metric spaces. Key results of their theory include $\operatorname{Ind} X = \dim X$ and a locally countable sum theorem. The dimension function ind is too weak to establish a satisfactory theory of dimension for metric spaces, and in 1962 P. Roy constructed an example of a metric space Δ such that $\operatorname{ind}\Delta = 0$ and $\operatorname{Ind}\Delta = 1$.

The breakthrough in dimension theory from separable metric to metric spaces seems to have left a lasting impression on Nagata. In his essay *Looking back at modern general topology in the last century* in [30], Nagata discusses the difference between classical general topology (old g.t.) and modern general topology (new g.t.). The emphasis in old g.t. was on separable metric and compact spaces; in new g.t., he states that "metrizability has taken over the position of separable metrizability, and paracompactness has almost taken over compactness' position." Nagata points out that this remarkable change was initiated by Stone's Theorem, closely followed by the Nagata–Smirnov and Bing Metrization Theorems and the subsequent development of dimension theory for general metric spaces.

We can summarize the goals of dimension theory as follows. Let $\mathcal{P}$ be a class of spaces that includes the separable metric spaces. Ideally, a dimension function d for $\mathcal{P}$, that is, a function $d : \mathcal{P} \to \{-1, 0, \dots, n, \dots, \infty\}$, should satisfy as many of the following properties as possible:

- (**subspace**) if $A \subseteq X$, then $d(A) \leqslant d(X)$ [at least for closed sets A];
- (**countable sum**) if X is the union of a countable collection $\mathcal{F}$ of closed sets such that $d(F) \leqslant n$ for all $F \in \mathcal{F}$, then $d(X) \leqslant n$;
- (**locally finite sum**) if X is the union of a locally finite collection $\mathcal{F}$ of closed sets such that $d(F) \leqslant n$ for all $F \in \mathcal{F}$, then $d(X) \leqslant n$;
- (**decomposition**) $d(X) \leqslant n$ if and only if X is the union of $n+1$ sets $A_1, \dots, A_{n+1}$ with $d(A_k) \leqslant 0$ for $1 \leqslant k \leqslant n+1$;
- (**addition**) if $A, B \subseteq X$, then $d(A \cup B) \leqslant d(A) + d(B) + 1$;
- (**product**) $d(X \times Y) \leqslant d(X) + d(Y)$;
- (**normalization**) $d(\emptyset) = -1$, $d(\{p\}) = 0$, and $d(\mathbb{R}^n) = n$;
- (**topological**) if $X, Y \in \mathcal{P}$ and are homeomorphic, then $d(X) = d(Y)$.

For example, each of ind, Ind and dim satisfy all of these properties for the class $\mathcal{P}$ of separable metric spaces and both Ind and dim satisfy all of these properties for the class $\mathcal{P}$ of metric spaces. See Engelking [37] for a discussion of various axioms for dimension.

Dimension theory is fortunate to have so many good books devoted to the subject, each with certain strengths and emphasis. The book by Hurewitz and Wallman is still the classic reference for separable metric spaces. Nagata's book [19,20] (also see his survey papers [21–28]) is the first successor to [45] and emphasizes the impressive breakthrough in dimension theory from separable metric to metric spaces. In addition, Nagata's book takes the first step in extending dimension theory beyond metric spaces. Nagami's book [56] covers the classical theory of dimension for metric spaces, but in addition emphasizes the considerable progress in metric-dependent dimension functions during the 1960's (due mainly to Nagami, Roberts and students of Roberts). In addition, there is an emphasis on dimension theory for normal spaces, compact spaces, and totally normal spaces (both dim and Ind) and there are many examples showing that certain classical properties of dimension break down outside the class of metric spaces. The book by Pears [61] is a continuation in this direction, with a strong emphasis on examples (for example, a complete discussion of Roy's space Δ and Filippov's example [39] of a compact space X with $\operatorname{ind} X < \operatorname{Ind} X$). Finally, Engelking [37] is a wonderful update of virtually all areas of dimension theory; there is a special emphasis on infinite-dimensional spaces, and the entire book is done in his careful and scholarly style with detailed references and extensive problem sets.

We recall a few basic definitions and notation. First of all, $\mathbb{I}^n$ is the product of n copies of the unit interval $[0,1]$ and $\mathbb{I}^\omega$ is a countable product of unit intervals with the product topology. Now let X be a set, let $p \in X$ and $A \subseteq X$, and let $\mathcal{G}$ and $\mathcal{H}$ be covers of X. The following notation is standard:

- $\operatorname{st}(p, \mathcal{G}) = \bigcup\{G\colon\ p \in G \in \mathcal{G}\}$;
- $\operatorname{st}(A, \mathcal{G}) = \bigcup\{G\colon\ G \in \mathcal{G} \text{ and } G \cap A \neq \emptyset\}$;
- $\mathcal{G} < \mathcal{H}$: $\mathcal{G}$ refines $\mathcal{H}$;
- $\mathcal{G} <^{\Delta} \mathcal{H}$: $\mathcal{G}$ is a *delta-refinement* of $\mathcal{H}$; in other words, $\{\operatorname{st}(p, \mathcal{G})\colon\ p \in X\} < \mathcal{H}$;
- $\mathcal{G} <^{*} \mathcal{H}$: $\mathcal{G}$ is a *star-refinement* of $\mathcal{G}$; in other words, $\{\operatorname{st}(G, \mathcal{G})\colon\ G \in \mathcal{G}\} < \mathcal{H}$.

A sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of X is a *development* for X if for each $p \in X$, $\{\operatorname{st}(p, \mathcal{G}_k)\colon\ k \geqslant 1\}$ is a local base for p. A base $\mathcal{B}$ for a space X is σ*-star-finite* if $\mathcal{B} = \bigcup \mathcal{B}_n$, where each $\mathcal{B}_n$ is a star-finite open cover of X (each element of $\mathcal{B}_n$ intersects at most a finite number of elements of $\mathcal{B}_n$). Spaces with a σ-star-finite base are intermediate between separable metric spaces and metric spaces and moreover the basic equation $\operatorname{ind} X = \operatorname{Ind} X$ extends to spaces with a σ-star-finite base. A metric ρ for X is an *ultrametric*, or *non-Archimedean*, if $\rho(x, z) \leqslant \max\{\rho(x, y), \rho(y, z)\}$ for all $x, y, z \in X$.

Let κ be an infinite cardinal. The *Baire space of weight* κ, denoted by $B(\kappa)$, is the zero-dimensional metric space constructed as follows: $B(\kappa)$ is the product of a countable number of copies of κ with ultrametric ρ defined by $\rho(\langle\alpha_n\rangle, \langle\beta_n\rangle) = 1/2^n$, where n is the smallest integer such that $\alpha_n \neq \beta_n$; $\rho(\langle\alpha_n\rangle, \langle\beta_n\rangle) = 0$ if $\alpha_n = \beta_n$ for all n. Every space with an ultrametric, and therefore every Baire space $B(\kappa)$, has a σ-star-finite base; moreover, $B(\kappa)$ has weight κ. Morita proved that $B(\kappa) \times \mathbb{I}^\omega$ is universal for all metric spaces of weight at most κ and having a σ-star finite base. Next, for each $\alpha \in \kappa$ let I_α be a copy of the unit interval $[0,1]$ and let $J(\kappa)$ be the set obtained from $\bigcup\{I_\alpha\colon\ \alpha \in \kappa\}$ by identifying all 0's. The set $J(\kappa)$ with metric defined by

$$\rho(x, y) = \begin{cases} |x - y| & \text{if } x, y \in I_\alpha, \\ x + y & \text{if } x \in I_\alpha,\ y \in I_\beta, \text{ and } \alpha \neq \beta, \end{cases}$$

is called the *star-space of weight* κ. For $\kappa > \omega$, $J(\kappa)$ is a metric space that does not have a σ-star finite base. The space $J(\kappa)^\omega$, the product of countably many copies of $J(\kappa)$, is universal for metric spaces of weight at most κ (Kowalsky [51]).

A space X is *totally normal* if X is normal and if every non-empty open subset U of X can be written as the union of a locally finite (in U) collection of open F_σ-subsets of X. Every perfectly normal space and every hereditarily paracompact Hausdorff space is totally normal and every totally normal space is completely normal.

With this background in place, we now turn to Nagata's many contributions to dimension theory, which we have divided into these six areas:

- characterizations of n-dimensional metric spaces in terms of a special base;
- characterizations of n-dimensional metrizable spaces in terms of a special metric;
- imbedding theorems and universal spaces for n-dimensional metric spaces;
- countable-dimensional metric spaces;
- dimension and rings of continuous functions;
- dimension theory beyond metric spaces.

1. Characterizations of *n*-dimensional metric spaces in terms of a special base

The following characterization of dimension plays a key role in the extension of dimension theory from separable metric to metric spaces: $\operatorname{Ind} X \leqslant n$ if and only if X has a σ-locally finite base $\mathcal{B}$ such that $\operatorname{Ind}(\overline{B} - B) \leqslant n - 1$ for all $B \in \mathcal{B}$. For example, it yields (by induction) an easy proof of the fact that if $A \subseteq X$ and $\operatorname{Ind} X \leqslant n$, then $\operatorname{Ind} A \leqslant n$. This characterization of dimension is obviously based on the Nagata–Smirnov Metrization Theorem and suggests the possibility of using other metrization theorems to characterize metric spaces of dimension $\leqslant n$ and more generally of characterizing dimension in terms of various base conditions. We now survey a number of such results obtained by Nagata in the papers [1,3,7,15].

Theorem 1.1. *([1,7]) A metric space X has* $\operatorname{Ind} X \leqslant n$ *if and only if there exist* $n + 1$ *countable sequences* $\mathcal{B}_{j,1}, \mathcal{B}_{j,2}, \ldots$ $(1 \leqslant j \leqslant n + 1)$ *such that*

(a) *for each sequence* $\mathcal{B}_{j,1}, \mathcal{B}_{j,2}, \ldots$ *and all* $k \geqslant 1$, $\mathcal{B}_{j,k}$ *is a pairwise disjoint open collection with* $\mathcal{B}_{j,k+1} < \mathcal{B}_{j,k}$;
(b) $\{B\colon B \in \mathcal{B}_{j,k}\colon 1 \leqslant j \leqslant n+1,\ k \geqslant 1\}$ *is a base for* X.

For example, a non-empty metric space X has $\operatorname{Ind} X = 0$ if and only if there is a base $\mathcal{B}$ for X that can be written as a countable union of disjoint open collections, say $\mathcal{B}_1, \mathcal{B}_2, \ldots$ such that $\mathcal{B}_{k+1}$ is a refinement of $\mathcal{B}_k$ for all $k \geqslant 1$.

Outline of the proof of Theorem 1.1. First prove the result for $n = 0$. Now assume $n \geqslant 0$. If $\operatorname{Ind} X \leqslant n$, write X as the union of $n+1$ spaces $X_1, \ldots, X_{n+1}$, each of dimension $\leqslant 0$. Apply the result for dimension 0 to each X_j to obtain a base $\mathcal{B}_{j,1}, \mathcal{B}_{j,2}, \ldots$ for X_j; by a suitable modification of each such sequence, one obtains the required $n+1$ collections that form a base for X. To prove the other direction, assume the $n+1$ sequences exist; use each sequence to construct a subspace of X of dimension zero and such that the union of these $n+1$ subspaces is X. $\square$

The Alexandroff–Urysohn Metrization Theorem characterizes the metrizability of a space X in terms of a development $\mathcal{G}_1, \mathcal{G}_2, \ldots$ for X such that $\mathcal{G}_{k+1} <^* \mathcal{G}_k$ for all $k \geqslant 1$. Nagata obtained three characterizations of dimension in terms of these metrizability conditions.

Theorem 1.2. *([1,3,7]) A* T_1*-space* X *is metrizable and has* $\operatorname{Ind} X \leqslant n$ *if and only if there is a development* $\mathcal{G}_1, \mathcal{G}_2, \ldots$ *for* X *such that* $\mathcal{G}_{k+1} <^* \mathcal{G}_k$ *for all* $k \geqslant 1$ *and such that one of the following three conditions holds for all* $k \geqslant 1$:

(a) *each element of* $\mathcal{G}_{k+1}$ *intersects at most* $n+1$ *elements of* $\mathcal{G}_k$;
(b) $\operatorname{ord} \mathcal{G}_k \leqslant n+1$;
(c) $\mathcal{G}_k$ *is a multiplicative cover of* X *of length at most* $n+1$.

A cover $\mathcal{G}$ of a set X is *multiplicative* if it has the following closure property: if H is non-empty and the intersection of a finite number of elements of $\mathcal{G}$, then $H \in \mathcal{G}$. Thus, if $G_1, \ldots, G_k \in \mathcal{G}$ and $G_1 \cap \cdots \cap G_k \neq \emptyset$, then $G_1, G_1 \cap G_2, \ldots, G_1 \cap \cdots \cap G_k$ are all in $\mathcal{G}$ and

$$G_1 \supseteq (G_1 \cap G_2) \supseteq \cdots \supseteq (G_1 \cap \cdots \cap G_k).$$

The *length* of a multiplicative cover $\mathcal{G}$ is the maximum number k such that there is a strictly decreasing sequence $G_1 \not\supseteq \cdots \not\supseteq G_k$ of k elements of $\mathcal{G}$. If $\mathcal{G}$ is multiplicative, then $\operatorname{length} \mathcal{G} \leqslant \operatorname{ord} \mathcal{G}$. More generally, if $\mathcal{G}$ is an open cover of X and $\mathcal{L}$ is the collection of all finite and non-empty intersections of elements of $\mathcal{G}$, then $\mathcal{L}$ is a multiplicative open cover of X and $\operatorname{length} \mathcal{L} \leqslant \operatorname{ord} \mathcal{G}$. These ideas are due to Alexandroff and Kolmogoroff. By Theorem 1.2, a non-empty T_1-space X is metrizable and has dimension zero if and only if there is a development $\mathcal{G}_1, \mathcal{G}_2, \ldots$ for X with $\mathcal{G}_{k+1} <^* \mathcal{G}_k$ for all $k \geqslant 1$ and one of the following holds for all $k \geqslant 1$: each collection $\mathcal{G}_k$ is pairwise disjoint; each element of $\mathcal{G}_{k+1}$ intersects at most one element of $\mathcal{G}_k$.

By far the hardest step in the proof of Theorem 1.2 is the implication (a) $\Rightarrow \operatorname{Ind} X \leqslant n$; for this Nagata uses Theorem 1.1 (among numerous other ideas). A clever use of the decomposition theorem is used to show that $\operatorname{Ind} X \leqslant n \Rightarrow$ (a). Comparatively speaking, the proofs that (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c) are fairly straightforward.

Nagata's work on base characterizations was continued by other researchers. For example, Nagami and Roberts [59] obtained a base characterization of dimension modeled on the Moore–Morita characterization of metrizability as follows: A T_1-space X is metrizable and has $\operatorname{Ind} X \leqslant n$ if and only if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of X with $\mathcal{G}_{k+1} < \mathcal{G}_k$ for all $k \geqslant 1$ and such that (a) for all $p \in X$, $\{\operatorname{st}^2(p, \mathcal{G}_k)\colon k \geqslant 1\}$ is a local base for p; (b) $\operatorname{ord} \mathcal{G}_k \leqslant n+1$ for all $k \geqslant 1$. Vaughan [64] proved that a metric space X has dimension $\leqslant n$ if and only if it has a σ-closure preserving base $\mathcal{B}$ such that $\operatorname{Ind}(\overline{B} - B) \leqslant n-1$ for all $B \in \mathcal{B}$.

In a 1963 paper Nagata gave yet another characterization of dimension in terms of base conditions. This time he used a completely new idea, the *rank* of a cover (introduced independently by Arhangel'skiĭ).

Theorem 1.3. *([15]) Let* X *be a metric space. Then* $\operatorname{Ind} X \leqslant n$ *if and only if* X *has a base* $\mathcal{B}$ *with* $\operatorname{rank} \mathcal{B} \leqslant n+1$.

Let $\mathcal{B}$ be a collection of subsets of X. Then $\operatorname{rank} \mathcal{B} \leqslant k$ if given $k+1$ distinct sets $B_1, \ldots, B_{k+1}$ in $\mathcal{B}$ such that $B_1 \cap \cdots \cap B_{k+1} \neq \emptyset$, there exist $i \neq j$ such that $B_i \subseteq B_j$. Note that $\operatorname{rank} \mathcal{B} \leqslant \operatorname{ord} \mathcal{B}$. Recall that a base $\mathcal{B}$ for a space X is *non-Archimedean* if whenever $A, B \in \mathcal{B}$ and $A \cap B \neq \emptyset$, then $A \subseteq B$ or $B \subseteq A$. Thus, the existence of a non-Archimedean base is equivalent to having a base of rank 1; moreover, Nagata's Theorem for $n = 0$ states that a metric space X has $\operatorname{Ind} X = 0$ if and only if it has a non-Archimedean base.

The concept of rank has been a very fruitful idea in metrization theory, compact spaces, and cardinal functions. For example, Arhangel'skiĭ proved the following: for a normal space X, $\dim X \leqslant n$ if and only if every finite open cover of X has a finite open refinement of rank $\leqslant n+1$; every compact Hausdorff space with a rank 1 base is metrizable. For additional related results and references, see [32,43,46].

Nagata's characterizations of dimension in terms of base conditions are interesting results in themselves and stimulated further research in topology, even beyond dimension theory. But in addition these results, especially Theorems 1.2(a) and 1.2(b), play an important role in his subsequent research on characterizing dimension in terms of special metrics and also on imbedding theorems. These two characterizations of dimension are used in conjunction as follows: *given* that $\operatorname{Ind} X \leqslant n$, there is an Alexandroff–Urysohn development for X as in (a); to *prove* that $\operatorname{Ind} X \leqslant n$, it suffices to construct an Alexandroff–Urysohn development for X that satisfies (b).

2. Characterizations of *n*-dimensional spaces in terms of a special metric

The dimension of a metrizable space X can be characterized in terms of the existence of a special metric on X that induces the topology of X. The first result of this type is due to de Groot [41] and asserts that for a metrizable space X, $\operatorname{Ind} X = 0$ if and only if there is an ultrametric ρ on X that induces the topology of X. Nagata obtained a number of characterizations of $\operatorname{Ind} X \leqslant n$ in terms of a special metric (see [2,3,7,11,14–16]), and most of these characterizations give de Groot's theorem for the case $n = 0$. In retrospect, it seems clear that Nagata's work in this area was motivated by the following observations.

- Given 4 points x, y_1, y_2, y_3 in $\mathbb{R}$, there exist $i \neq j$ such that $|y_i - y_j| \leqslant |x - y_j|$.
- Higher dimension versions of the above fail. For example, consider $\mathbb{R}^2$ with Euclidean metric ρ and the five points $x = \langle 0, 0\rangle$, $y_1 = \langle 1, 0\rangle$, $y_2 = \langle 0, 1\rangle$, $y_3 = \langle -1, 0\rangle$, and $y_4 = \langle 0, -1\rangle$; $\rho(y_i, y_j) \leqslant \rho(x, y_j)$ fails for all $i \neq j$.

Let us list several conditions on a metric ρ that are related to these observations:

$(0)_n$ given $\epsilon > 0$ and $n+3$ points $x, y_1, \dots, y_{n+2}$ in X with $\rho(B_\rho(x, \epsilon/2), y_j) < \epsilon$ for $1 \leqslant j \leqslant n+2$, there exists i, j with $i \neq j$ such that $\rho(y_i, y_j) < \epsilon$;
$(1)_n$ given $n+3$ points $x, y_1, \dots, y_{n+2}$ in X, there exist i, j with $i \neq j$ such that $\rho(y_i, y_j) \leqslant \rho(x, y_j)$;
$(2)_n$ given $n+3$ points $x, y_1, \dots, y_{n+2}$ in X, there exist i, j, k with $i \neq j$ such that $\rho(y_i, y_j) \leqslant \rho(x, y_k)$.

Nagata's first result on special metrics is the following:

Theorem 2.1. *([2,7]) For a metrizable space X, $\operatorname{Ind} X \leqslant n$ if and only if there is a metric ρ for X that satisfies $(0)_n$ and induces the topology of X.*

Outline of proof. Assuming $\operatorname{Ind} X \leqslant n$, use the development given in Theorem 1.2(a) to construct the required metric ρ (reminiscent of, but considerably more difficult than, Frink's construction of a metric from the conditions of the Alexandroff–Urysohn Metrization Theorem). Assuming the existence of a metric ρ that satisfies $(0)_n$, use Theorem 1.2(b) to show $\operatorname{Ind} X \leqslant n$.

The condition $(0)_n$ for $n = 0$ gives de Groot's result on the existence of an ultrametric. More precisely, Nagata proves the following: given a metric ρ on X, the following are equivalent: (1) ρ is an ultrametric; (2) given $\epsilon > 0$ and $x, y_1, y_2 \in X$, if $\rho(B_\rho(x, \epsilon/2), y_1) < \epsilon$ and $\rho(B_\rho(x, \epsilon/2), y_2) < \epsilon$, then $\rho(y_1, y_2) < \epsilon$. On the other hand, the Euclidean metric on $\mathbb{R}$ fails to satisfy $(0)_n$ for the case $n = 1$. □

In [7] Nagata points out that the proof of Theorem 2.1 gives a characterization of dimension for compact metrizable spaces in terms of the following variation of $(1)_n$: $\operatorname{ind} X \leqslant n$ if and only if there is a metric ρ on X that induces the topology of X such that given $\epsilon > 0$ and $n+3$ points $x, y_1, \dots, y_{n+2}$ in X such that $\rho(x, y_j) < \epsilon$ for $1 \leqslant j \leqslant n+2$, there exist i, j with $i \neq j$ such that $\rho(y_i, y_j) < \epsilon$.

In 1956 Nagata gave a second generalization of de Groot's zero-dimensional characterization as follows. Call a function $\rho : X \times X \to [0, \infty)$ a *non-Archimedean parametric* if it satisfies these three conditions:

(1) $\rho(x, y) = \rho(y, x)$;
(2) $\{y\colon\ \rho(x, y) < \epsilon\}$ is an open set for all $\epsilon > 0$;
(3) $\rho(x, z) \leqslant \max\{\rho(x, y), \rho(y, z)\}$.

Theorem 2.2. *([3,7]) For a metrizable space X, $\operatorname{Ind} X \leqslant n$ if and only if there exist $n+1$ non-Archimedean parametrics $\rho_1, \dots, \rho_{n+1}$ on X such that the following is a metric for X compatible with the topology of X:*

$$\rho(x, y) = \inf\bigl\{\rho_0(x, z_1) + \rho_0(z_1, z_2) + \cdots + \rho_0(z_r, y)\colon\ z_1, \dots, z_r \in X\bigr\}$$

where

$$\rho_0(x, y) = \min\bigl\{\rho_1(x, y), \dots, \rho_{n+1}(x, y)\bigr\}.$$

For $n = 0$, the metric ρ is an ultrametric and thus Theorem 2.2 for $n = 0$ gives de Groot's Theorem.

Outline of proof of Theorem 2.2. $\operatorname{Ind} X \leqslant n \Rightarrow$ existence of metric: use the decomposition theorem and construct the parametrics on the $n+1$ zero-dimensional subspaces; existence of metric $\Rightarrow \operatorname{Ind} X \leqslant n$: use the $n+1$ parametrics to construct subspaces $X_1, \ldots, X_{n+1}$ whose union is X; use the base characterization in Theorem 1.1 to show that each has $\operatorname{Ind} \leqslant 0$. □

In [42], de Groot uses Nagata's Theorem 2.1 to prove that for a separable metrizable space X, $\operatorname{ind} X \leqslant n$ if and only if there is a totally bounded metric ρ on X that induces the topology of X and satisfies $(2)_n$. He emphasizes that his result applies only for the separable case; the question of whether it extends to arbitrary metrizable spaces is apparently still unsolved. Note the following properties of $(2)_n$:

- any metric ρ that satisfies $(2)_n$ for $n = 0$ is an ultrametric;
- the Euclidean metric on $\mathbb{R}$ satisfies $(2)_n$ for $n = 1$ (not so for Nagata's metric in Theorem 2.1);
- the Euclidean metric on $\mathbb{R}^2$ fails to satisfy $(2)_n$ for $n = 2$.

Theorem 2.3 below is Nagata's final result in this general direction. It is an improvement over Theorem 2.1 and can be considered a partial solution to de Groot's question on characterization by $(2)_n$. For $n = 0$, the metric ρ in Theorem 2.3 has the property that $\rho(x, z) \leqslant \rho(x, y)$ or $\rho(x, z) \leqslant \rho(y, z)$, and this is equivalent to ρ being an ultrametric. Thus Theorem 2.3, like Theorems 2.1 and 2.2, extends de Groot's Theorem to higher dimension. Theorem 2.3 was obtained independently by Ostrand.

Theorem 2.3. *([14,16]) For a metrizable space X, $\operatorname{Ind} X \leqslant n$ if and only if there is a metric ρ on X that satisfies $(1)_n$ and induces the topology of X.*

Every metric ρ that satisfies $(1)_n$ also satisfies $(2)_n$, and therefore Theorem 2.3 tells us that if X is metrizable with $\operatorname{Ind} X \leqslant n$, then there is a metric ρ on X that induces the topology of X and satisfies $(2)_n$. Thus de Groot's question comes down to proving that the existence of a metric ρ on X that satisfies $(2)_n$ implies $\operatorname{Ind} X \leqslant n$.

In addition to the above results motivated by de Groot, Nagata obtained two other characterizations of dimension in terms of a metric.

Theorem 2.4. *([15]) For a metrizable space X, $\operatorname{Ind} X \leqslant n$ if and only if there is a metric ρ on X that induces the topology of X such that for all $\epsilon > 0$:*

(1) *the boundary of each spherical ball $B_\rho(x, \epsilon)$ has $\operatorname{Ind} \leqslant n - 1$;*
(2) *$\{B_\rho(x, \epsilon)\colon x \in X\}$ is closure preserving.*

The closure preserving requirement is somewhat restrictive, and so Nagata derives from Theorem 2.4 the following

Corollary. *For a metrizable space X, $\operatorname{Ind} X \leqslant n$ if and only if there is a metric ρ on X that induces the topology of X and such that for every closed subset H of X and all $\epsilon > 0$,*

$$\operatorname{Ind}\big[\overline{B_\rho(H, \epsilon)} - B_\rho(H, \epsilon)\big] \leqslant n - 1.$$

See the survey paper [29] by Hattori and Nagata for an up-to-date discussion of special metrics.

3. Imbedding theorems and universal spaces for *n*-dimensional metric spaces

Urysohn's proof that every regular space with a countable base is metrizable also shows that every separable metric space can be imbedded in $[0, 1]^\omega$. For spaces of dimension $\leqslant n$, there is a sharper result: every separable metric space of dimension $\leqslant n$ can be imbedded in $[0, 1]^{2n+1}$, the product of $2n + 1$ copies of $[0, 1]$ (due to Menger). Moreover, this result is the best possible; from graph theory we know that K_5, the complete graph on 5 vertices, and $K_{3,3}$, the "utility graph," are one-dimensional spaces that cannot be imbedded in the plane. In the papers [3,4,7] Nagata obtains imbedding theorems for metric spaces that extend the above-mentioned results on separable metric spaces.

Theorem 3.1. *([3,7]) Every metric space of dimension $\leqslant n$ can be embedded in the product of $n + 1$ metric spaces, each of dimension at most 1.*

Theorem 3.2. *([4]) Every metric space can be embedded in the product of a countable number of metric spaces, each of dimension at most 1.*

Theorem 3.1 is the best possible; S^2 cannot be embedded in the product of two one-dimensional spaces (Borsuk, 1975). We briefly remark on the proof of Theorem 3.1. First of all, Nagata uses his Theorems 1.2(a) and 1.2(b) that characterize dimension in terms of a development. Let $\langle X, \rho\rangle$ be a metric space of dimension n; Nagata uses the development given in Theorem 1.2(a) to construct $n+1$ spaces $F_1(X), \dots, F_{n+1}(X)$, uses Theorem 1.2(b) to prove that each is metrizable and has dimension $\leqslant 1$, and then imbeds X in $F(X_1) \times \cdots \times F(X_{n+1})$. These constructions are quite ingenious and involved. The overall plan of attack for Theorem 3.2 is similar but technically more difficult.

Imbedding theorems give rise to questions of universality. Let $\mathcal{P}$ be a class of spaces and let $X \in \mathcal{P}$. We say that X is *universal* for $\mathcal{P}$ if every $Y \in \mathcal{P}$ is homeomorphic to a subspace of X. For example, $\mathbb{I}^\omega$ is universal for separable metric spaces. The following classical theorem due to Nöbeling is a sharper version of Menger's result that every separable metric space of dimension $\leqslant n$ imbeds in $\mathbb{I}^{2n+1}$: the space N_n^{2n+1} is universal for separable metric spaces of dimension $\leqslant n$, where

$$N_n^{2n+1} = \{x\colon\ x \in \mathbb{I}^{2n+1} \text{ and } x \text{ has at most } n \text{ rational coordinates}\}.$$

Thus, for $n = 0$ the set of irrationals in $[0,1]$ is universal for all zero-dimensional separable metric spaces. Nagata extended Nöbeling's result to metric spaces and also to metric spaces with a σ-star-finite base (also see [12]).

Theorem 3.3. *([13]) Let κ be an infinite cardinal. The space $N(\kappa, n)$ is universal for n-dimensional metric spaces of weight at most κ, where*

$$N(\kappa, n) = \{x\colon\ x \in J(\kappa)^\omega \text{ and at most } n \text{ non-zero coordinates of } x \text{ are rational}\}.$$

The space $N(\kappa, n)$ is often referred to as *Nagata's universal n-dimensional metric space*.

Theorem 3.4. *([7]) Let κ be an infinite cardinal. The space $B(\kappa) \times N_n^{2n+1}$ is universal for n-dimensional metric spaces with a σ-star-finite base and weight at most κ.*

4. Countable-dimensional metric spaces

Let X be a metric space. If the inequality $\dim X \leqslant n$ fails for all $n \geqslant -1$, then X is said to be *infinite-dimensional*. Nagata's goal in [6,8,10,15] and the survey paper [25] is to extend the theory of finite-dimensional metric spaces to infinite-dimensional metric spaces. To see that the problem is by no means straightforward, consider the following three examples of infinite-dimensional separable metric spaces:

- $\mathbb{I}^\omega = [0,1]_1 \times \cdots \times [0,1]_n \times \cdots$ (countably infinite product of unit intervals);
- $R^\omega = \{x\colon\ x \in \mathbb{I}^\omega$ and x has at most a finite number of rational coordinates$\}$;
- $K^\omega = \{x\colon\ x \in \mathbb{I}^\omega$ and x has at most a finite number of non-zero coordinates$\}$.

$\mathbb{I}^\omega$ is a compact metric space that cannot be written as a countable union of finite-dimensional subspaces (proof later). On the other hand, the subspace R^ω of $\mathbb{I}^\omega$ can be so written, namely

$$R^\omega = \bigcup_{n \geqslant 1} [0,1]_1 \times \cdots \times [0,1]_n \times P \times P \times \cdots,$$

where P is the set of irrationals in $[0,1]$. But R^ω cannot be written as countable union of finite-dimensional *closed* subspaces (proved by Nagata in [8]). The subspace K^ω of $\mathbb{I}^\omega$ is infinite-dimensional and moreover can be written as a countable union of closed finite-dimensional subspaces, namely

$$K^\omega = \bigcup_{n \geqslant 1} [0,1]_1 \times \cdots \times [0,1]_n \times \{0\} \times \{0\} \times \cdots.$$

These examples suggest the following definitions. A metric space X is *countable-dimensional* if it can be written as a countable union of zero-dimensional subspaces. By the decomposition theorem (every metric space of dimension n can be written as the union of $n+1$ zero-dimensional subspaces), this is equivalent to: X can be written as a countable union of finite-dimensional subspaces. A metric space X is *strongly countable-dimensional* if it can be written as a countable union of closed finite-dimensional subspaces. Thus, $\mathbb{I}^\omega$ is not countable-dimensional, R^ω is countable-dimensional but not strongly countable-dimensional, and K^ω is strongly countable-dimensional but not finite-dimensional. Nagata made many important and significant contributions to the area of infinite-dimensional spaces for both separable metric and metric spaces; these contributions can be divided into three categories: **characterizations**, **universal spaces**, **special metrics**.

4.1. Characterizations

A first step in establishing a theory of countable-dimensional metric spaces is to give characterizations in terms of special bases or related conditions.

Theorem 4.1. *([6,8]) Let X be a metric space. The following are equivalent*:

(1) *X is countable-dimensional*;
(2) *there is a sequence $\mathcal{B}_1, \mathcal{B}_2, \dots$ of locally finite open covers of X such that $\mathcal{B} = \bigcup \mathcal{B}_n$ is a base for X and each $x \in X$ is in at most a finite number of elements of $\{\overline{B} - B\colon B \in \mathcal{B}\}$*;
(3) *given a collection $\{F_\alpha\colon \alpha < \kappa\}$ of closed subsets of X and a corresponding collection $\{U_\alpha\colon \alpha < \kappa\}$ of open sets such that $F_\alpha \subseteq U_\alpha$ and $\{U_\beta\colon \beta < \alpha\}$ is locally finite for all $\alpha < \kappa$, there is a collection $\{V_\alpha\colon \alpha < \kappa\}$ of open sets with $F_\alpha \subseteq V_\alpha \subseteq U_\alpha$ for all $\alpha < \kappa$ and such that each $x \in X$ is in at most a finite number of elements of $\{\overline{V}_\alpha - V_\alpha\colon \alpha < \kappa\}$.*

Theorem 4.2. *([6,8]) A metric space X is countable-dimensional if and only if there is a countable sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ of locally finite closed covers of X such that*:

(a) *if U is an open neighborhood of $x \in X$, then there exists $n \geqslant 1$ such that $\mathrm{st}(x, \mathcal{F}_n) \subseteq U$*;
(b) *for all $n \geqslant 1$, $\mathcal{F}_n = \{F(\alpha_1, \dots, \alpha_n)\colon \alpha_1, \dots, \alpha_n \in \kappa\}$ ($F(\alpha_1, \dots, \alpha_n) = \emptyset$ allowed)*;
(c) *$F(\alpha_1, \dots, \alpha_n) = \bigcup \{F(\alpha_1, \dots, \alpha_n, \beta)\colon \beta \in \kappa\}$*;
(d) *for all $x \in X$, $\sup\{\mathrm{ord}_x \mathcal{F}_n\colon n \geqslant 1\}$ is finite, where $\mathrm{ord}_x \mathcal{F}_n$ is the number of elements of $\mathcal{F}_n$ that contain x.*

Nagata uses his Theorem 4.2 to obtain a characterization of countable-dimensional spaces in terms of a closed map.

Theorem 4.3. *([6,8]) A metric space X of weight κ is countable-dimensional if and only if there is a subset S of $B(\kappa)$ and a closed and continuous function f from S onto X such that for all $x \in X$, $f^{-1}(x)$ is finite.*

The Eilenberg–Otto characterization of dimension states that a metric space X has dimension $\leqslant n$ if and only if given $n+1$ pairs $A_1, B_1; \dots; A_{n+1}, B_{n+1}$ of disjoint closed sets in X, there is a sequence $L_1, \dots, L_{n+1}$ of closed sets such that L_k separates A_k and B_k for $1 \leqslant k \leqslant n+1$ and $\bigcap L_k = \emptyset$. In [10], Nagata obtained an analogue of this result for countable-dimensional spaces that can be stated as follows:

Theorem 4.4. *([10]) A metric space X is countable-dimensional if and only if given a countable sequence $A_1, B_1; A_2, B_2; \dots$ of pairs of disjoint closed sets, there is a sequence $L_1, L_2, \dots$ of closed sets such that L_k separates A_k and B_k for all $k \geqslant 1$ and every point of X is in at most a finite number of elements of $\{L_n\colon n \geqslant 1\}$.*

The Eilenberg–Otto characterization of dimension can be used to show that $\operatorname{Ind} \mathbb{I}^n \geqslant n$. In much the same way, Theorem 4.4 can be used to show that $\mathbb{I}^\omega$ is not countable-dimensional. To see this, for $k \geqslant 1$ let A_k and B_k be the subsets of $\mathbb{I}^\omega$ determined by the equations $x_k = 0$ and $x_k = 1$ respectively (opposite faces of $\mathbb{I}^\omega$). Now suppose by way of contradiction that $\mathbb{I}^\omega$ is countable-dimensional. By Theorem 4.4, there is a sequence $L_1, L_2, \dots$ of closed sets such that L_k separates A_k and B_k for all $k \geqslant 1$ and every point of X is in at most a finite number of elements of $\{L_n\colon n \geqslant 1\}$. On the other hand, it is a consequence of the Brouwer Fixed Point Theorem for $\mathbb{I}^\omega$ that $\bigcap L_n \neq \emptyset$.

Theorem 4.5. *([8]) Let X be a metric space. Then X is strongly countable-dimensional if and only if there is a development $\mathcal{G}_1, \mathcal{G}_2, \dots$ for X with $\mathcal{G}_{k+1} <^* \mathcal{G}_k$ for all $k \geqslant 1$ and such that for each $x \in X$, there is a positive integer n_x such that for all $k \geqslant 1$, x is in at most n_x elements of $\mathcal{G}_k$.*

The next result is a nice application of Nagata's Theorem 1.3 that characterizes dimension in terms of the rank of a base.

Theorem 4.6. *([15]) A metric space X is strongly countable-dimensional if and only if X has a base $\mathcal{B}$ such that for all $x \in X$, $\mathrm{rank}_x \mathcal{B}$ is finite (there is a positive integer n_x such that if $x \in B_1 \cap \dots \cap B_r$, where $B_1, \dots, B_r \in \mathcal{B}$ and $r > n_x$, then $B_i \subseteq B_j$ for some $i \neq j$).*

4.2. Universal spaces

Another direction of research is to find universal spaces for countable and strongly countable-dimensional spaces. First of all, there are these "classical" results: $\mathbb{I}^\omega$ is universal for all separable metric spaces (Urysohn); $B(\kappa) \times \mathbb{I}^\omega$ is universal for all metric spaces of weight at most κ and having a σ-star-finite base (Morita); $J(\kappa)^\omega$ is universal for all metric spaces of weight at most κ (Kowalsky). In the papers [6,8,13] Nagata obtains a number of universality results that can be added to this list.

- R^ω is universal for countable-dimensional separable metric spaces;
- K^ω is universal for strongly countable-dimensional separable metric spaces (obtained independently by Smirnov);
- $B(\kappa) \times R^\omega$ is universal for countable-dimensional metrics spaces of weight at most κ and having a σ-star-finite base;
- $B(\kappa) \times K^\omega$ is universal for strongly countable-dimensional metrics spaces of weight at most κ and having a σ-star-finite base.

- $N(\kappa)$ is universal for countable-dimensional metric spaces of weight at most κ, where

$$N(\kappa) = \{x\colon\ x \in J(\kappa)^\omega \text{ and at most a finite number of non-zero coordinates of } x \text{ are rational}\} = \bigcup_{n\geqslant 1} N(\kappa, n).$$

The missing link was obtained in a 1990 paper by Olszewski [60], where he constructs a subspace of $J(\kappa)^\omega \times \mathbb{I}^\omega$ that is universal for strongly countable-dimensional metrics spaces of weight at most κ.

4.3. Special metrics

Nagata's survey paper [25] includes some original contributions on special metrics for strongly countable-dimensional metrizable spaces. He begins by defining infinite analogues of the two conditions $(1)_n$ and $(2)_n$:

$(1)_\infty$ for all $x \in X$, there is $n(x) \geqslant 0$ such that given $y_1, \ldots, y_{n(x)+2} \in X$, there exist i, j with $i \neq j$ such that $\rho(y_i, y_j) \leqslant \rho(x, y_j)$;
$(2)_\infty$ for all $x \in X$, there is $n(x) \geqslant 0$ such that given $y_1, \ldots, y_{n(x)+2} \in X$, there exist i, j, k with $i \neq j$ such that $\rho(y_i, y_j) \leqslant \rho(x, y_k)$.

Theorem 4.7. *The following are equivalent for a metrizable space X:*

(1) *X is strongly countable-dimensional*;
(2) *there is a metric ρ compatible with the topology of X that satisfies* $(1)_\infty$;
(3) *there is a metric ρ compatible with the topology of X that satisfies* $(2)_\infty$.

Finally, Nagata considers two other possibilities for characterizing countable-dimensional spaces in terms of a special metric, namely:

$(3)_\infty$ given $x \in X$ and an infinite sequence $\langle y_n \rangle$ of distinct points of X, there exist i, j with $i \neq j$ such that $\rho(y_i, y_j) \leqslant \rho(x, y_j)$;
$(4)_\infty$ given $x \in X$ and an infinite sequence $\langle y_n \rangle$ of distinct points of X, there exist i, j, k with $i \neq j$ such that $\rho(y_i, y_j) \leqslant \rho(x, y_k)$.

Nagata eliminates the possibility of $(4)_\infty$ by showing that every metrizable space has a metric ρ compatible with the topology of X that satisfies $(4)_\infty$.

5. Rings of continuous functions

A major theme of rings of continuous functions is the study of the interplay between topological properties of X and algebraic properties of $C(X)$, the ring of all continuous real-valued bounded functions from X into $\mathbb{R}$. For example, a compact space can be completely characterized in terms of the ideals of $C(X)$. Henceforth we assume that X is a metric space. In [48] Katětov defined the notion of the analytical dimension of $C(X)$, denoted by $\dim C(X)$, and proved that $\operatorname{Ind} X = \dim C(X)$ for every compact metric space X.

Now let $U(X)$ be the subset of $C(X)$ consisting of the uniformly continuous functions; $U(X) = C(X)$ in the case where X is compact. In a 1960 paper [9], Nagata extended Katětov's result to locally compact metric spaces as follows.

Theorem 5.1. *Let X be a locally compact metric space. Then* $\operatorname{Ind} X = \dim(U(X), C(X))$. *More precisely*:

- *if X is metric, then* $\operatorname{Ind} X \leqslant \dim(U(X), C(X))$;
- *if X is locally compact metric, then* $\dim(U(X), C(X)) \leqslant \operatorname{Ind} X$.

Here are the required definitions. A commutative ring R with multiplicative identity e is said to be *analytical* if R is a topological ring and in addition has a continuous real-valued scalar multiplication. For example, the ring $C(X)$ is an analytical ring with topology given by the metric

$$\rho(f, g) = \sup\{|f(x) - g(x)|\colon\ x \in X\}.$$

A subring S of an analytical ring R is *analytically closed* if it satisfies

(1) $\lambda e \in S$ for every scalar $\lambda \in \mathbb{R}$;
(2) for all $x \in R$, if $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, where $a_1, \ldots, a_n \in S$, then $x \in S$;
(3) S is a closed set.

Now let B and U be subsets of an analytical ring R. Then B is an *analytical base* for U in R if for every analytically closed subring S of R, if $B \subseteq S$, then $U \subseteq S$. The *analytical dimension* of U, denoted by $\dim(U, R)$, is the smallest cardinality of an analytical base B for U. In the case where X is compact and the analytical ring is $C(X)$ we have $C(X) = U(X)$, $\dim(U(X), C(X)) = \dim C(X)$, and Katětov's result follows from Nagata's Theorem.

Proofs of Katětov's result have appeared in several books: both of Nagata's books on dimension theory, the book by Pears, and Gillman and Jerison [40]. Chapter 10 of Pears gives the most modern and up to date treatment of the relationship between algebras of continuous functions on a space X and the dimension of X.

6. Dimension theory beyond metric spaces

Nagata's main interests in dimension theory focused on metric spaces. Nevertheless, he also made valuable contributions to the theory for more general classes of spaces. These results fall into four areas: dimension-raising closed mappings; product theorems; characterization of dim in normal spaces; cardinal invariants and dimension theory.

6.1. Dimension-raising closed mappings

Let X and Y be topological spaces and let f be a function from X onto Y. A fundamental problem in dimension theory is to investigate the relationship between the dimension of X, the dimension of Y, and properties of f. Results of this type are often classified as *dimension raising* or *dimension lowering*. Peano showed that an interval $[0, 1]$ can be mapped continuously onto a square $[0, 1] \times [0, 1]$; thus, a continuous closed mapping can raise dimension. In a 1930 paper Hurewitz [44] proved the following result on such dimension-raising maps: Let X and Y be separable metric spaces and let f be a continuous and closed mapping from X onto Y such that for all $y \in Y$, $f^{-1}(y)$ has at most $m + 1$ points. Then $\operatorname{Ind} Y \leqslant \operatorname{Ind} X + m$. To illustrate:

- if f is a continuous mapping from $[0, 1]$ onto a separable metric space Y and f is at most two-to-one, then $\operatorname{Ind} Y \leqslant 2$.
- if f is a continuous mapping from the Cantor set onto the cube $[0, 1]^3$, then $f^{-1}(y)$ has at least 4 points for some $y \in [0, 1]^3$.

In a 1941 paper Roberts [62] proved that the Hurewitz result is the best possible for separable metric spaces: given $0 \leqslant m \leqslant n$, there exist separable metric spaces X and Y such that $\operatorname{Ind} X = n - m$, $\operatorname{Ind} Y = n$, and there is a continuous closed f from X onto Y such that $f^{-1}(y)$ consists of at most $m + 1$ points. Nagami [57] extended Roberts' result to metric spaces.

Hurewitz's original result has been the source of considerable research in dimension theory beyond the two results just mentioned; for a comprehensive survey, see Pears. An early result due to Morita [55] extends the Hurewitz inequality by assuming that X and Y are both normal and proving $\dim Y \leqslant \operatorname{Ind} X + m$. Nagata generalized Hurewitz's Theorem in a different way as follows:

Theorem 6.1. *([5]) Let f be a continuous and closed mapping from a normal space X onto a perfectly normal space Y such that for all $y \in Y$, the boundary of $f^{-1}(y)$ has at most $m + 1$ points. Then* $\operatorname{Ind} Y \leqslant \operatorname{Ind} X + m$.

Nagata's proof actually holds for Y hereditarily normal (see his 1965 book [19]). Moreover, in his 1983 book [20], he proves the following variation of this result:

Theorem 6.2. *Let f be a continuous and closed mapping from a totally normal space X onto a normal space Y such that for all $y \in Y$, the boundary of $f^{-1}(y)$ has at most $m + 1$ points. Then* $\operatorname{Ind} Y \leqslant \operatorname{Ind} X + m$.

Here are two other related results of interest. Let f be a continuous and closed mapping from X onto Y. (1) If X and Y are both normal and $f^{-1}(y)$ consists of at most $m + 1$ points for all $y \in Y$, then $\dim Y \leqslant \dim X + m$ (see [66]); (2) if X and Y are metric spaces and f is exactly k-to-one, then $\operatorname{Ind} X = \operatorname{Ind} Y$ (see [63]).

6.2. Product theorems

A basic result in dimension theory for metric spaces is the product theorem

$$\operatorname{Ind}(X \times Y) \leqslant \operatorname{Ind} X + \operatorname{Ind} Y.$$

Since $\operatorname{Ind} X = \dim X$ for all metric spaces, this is equivalent to $\dim(X \times Y) \leqslant \dim X + \dim Y$. However, extending either result beyond metric spaces is a challenge. For example, Filippov [38] constructed compact Hausdorff spaces X and Y such that $\operatorname{Ind} X = 1$, $\operatorname{Ind} Y = 2$, and $\operatorname{Ind}(X \times Y) \geqslant 4$. In 1978 Wage [65] constructed two examples as follows: there is a separable metric space X and a paracompact space Y such that $\dim X = \dim Y = 0$ and $\dim(X \times Y) = 1$; under CH, there is a locally compact perfectly normal space X with $\dim X = 0$ such that $X \times X$ is perfectly normal and $\dim X \times X > 0$.

In a 1967 paper Nagata obtained a number of product theorems for Ind, and for this he introduces the following new idea: Given spaces X and Y, we say that $X \times Y$ is an *F-product* if given any pair of disjoint closed subsets H and K of $X \times Y$, there is a collection $\{F_t\colon\ t \in T\}$ of closed rectangles in $X \times Y$ that covers $X \times Y$ and a corresponding σ-locally finite collection $\{U_t\colon\ t \in T\}$ of open rectangles in $X \times Y$ with $F_t \subseteq U_t$ for all $t \in T$ such that $H \cap U_t = \emptyset$ or $K \cap U_t = \emptyset$ for all $t \in T$. Nagata's main result is the following.

Theorem 6.3. *([17]) Let X and Y be spaces ($X \neq \emptyset$ or $Y \neq \emptyset$) such that $\operatorname{Ind} X \leqslant n$ and $\operatorname{Ind} Y \leqslant m$. If $X \times Y$ is a totally normal F-product, then $\operatorname{Ind} X \times Y \leqslant n + m$.*

The proof is by induction on $n + m$. Under what conditions on X and Y is $X \times Y$ an F-product? Nagata proves the following.

- If X and Y are normal M'-spaces, then $X \times Y$ is an F-product. (X is an *M'-space* if there is a closed, continuous function f from X onto a metric space Y such that for all $y \in Y$, $f^{-1}(y)$ is compact; this is slightly stronger than the definition of an M-space, which only requires that $f^{-1}(y)$ be countably compact.)
- If X is a normal P-space and Y is metric, then $X \times Y$ is an F-product. P-spaces are due to Morita; every perfectly normal space is a P-space.
- If X is a locally compact paracompact space and Y is paracompact, then $X \times Y$ is an F-product.

Thus, if $X \times Y$ is totally normal, then the product theorem $\operatorname{Ind} X \times Y \leqslant \operatorname{Ind} X + \operatorname{Ind} Y$ holds in each of the following cases:

- X is perfectly normal and Y is metric;
- X is compact and Y is paracompact;
- X and Y are paracompact M-spaces.

See the book by Pears for a detailed discussion of Nagata's theorem and a survey of related results; we mention two. Nagami proved [58] that if X is a Hausdorff P-space, Y a Σ-space, and $X \times Y$ is hereditarily paracompact, then $\operatorname{Ind} X \times Y \leqslant \operatorname{Ind} X + \operatorname{Ind} Y$. Pasynkov [61] announced that if X and Y are totally normal spaces such that $X \times Y$ is paracompact and an F-product, then $\operatorname{Ind} X \times Y \leqslant \operatorname{Ind} X + \operatorname{Ind} Y$.

6.3. A characterization of covering dimension for normal spaces

In [18,24] Bruijning and Nagata give a characterization of dimension that is motivated by a much earlier result due to Pontrjagin and Schnirelmann (1932). The basic idea of Pontrjagin–Schnirelmann is that the dimension of a compact metric space X can be calculated in terms of the relationship between the positive number ϵ and the least number $k(\epsilon)$ of ϵ-small sets that are required to cover X (a set is *ϵ-small* if it has diameter $\leqslant \epsilon$). A *simplified* version of their formula states that

$$\dim X = -\lim_{\epsilon \to 0} \frac{\log k(\epsilon)}{\log \epsilon}.$$

We illustrate with an example. Let X be the unit square $[0,1] \times [0,1]$. If $\epsilon = \sqrt{2}/n$, then $k(\epsilon) = n^2$ and we have

$$\dim X = -\lim_{n \to \infty} \frac{\log n^2}{\log(\sqrt{2}/n)} = 2.$$

The above formula is sometimes used as a method of assigning a fractal dimension to irregular spaces; for example, the Cantor set has fractal dimension $\log 2/\log 3$ and the Koch curve has fractal dimension $\log 4/\log 3$. There is a complete proof of the Pontrjagin–Schnirelmann Theorem in the second edition of Nagata's book on dimension theory. See [33] for a discussion of the relationship between the precise version of the Pontrjagin–Schnirelmann result and the Hausdorff–Besicovitch dimension.

The goal of [18] is to extend these ideas beyond the setting of metric spaces. To simplify the discussion of their main result, we henceforth assume that X is an infinite normal space (their results actually extends to Tychonoff spaces). For each $k \geqslant 1$ let $\Delta_k(X)$ be the smallest positive integer such that for every open cover $\mathcal{U}$ of X of cardinality at most k, there is an open cover $\mathcal{V}$ of X of cardinality at most $\Delta_k(X)$ such that $\mathcal{V} <^{\Delta} \mathcal{U}$. Intuitively speaking, $\mathcal{V} <^{\Delta} \mathcal{U}$ captures the idea that the cover $\mathcal{V}$ is $\mathcal{U}$-small. Here is their main result:

Main Theorem. *Let X be an infinite normal space with $\dim X = n$ and let $k \geqslant 1$. Then*

$$\Delta_k(X) = 2^k - 1 \quad \text{if } k \leqslant n+1;$$
$$\Delta_k(X) = \binom{k}{1} + \cdots + \binom{k}{n+1} \quad \text{if } k \geqslant n+1.$$

The proof uses the non-trivial lemma that if X is normal, then $\dim X \leqslant n$ if and only if for every open cover $\mathcal{U} = \{U_1, \ldots, U_{n+2}\}$ of X, there is an open cover $\mathcal{V} = \{V_1, \ldots, V_{n+2}\}$ of X with $V_j \subseteq U_j$ for $1 \leqslant j \leqslant n+2$ and $\bigcap V_j = \emptyset$. Moreover, a special case of the proof of the Main Theorem shows that if $\mathcal{U}$ is any finite open cover of a normal space X, then there is a finite open cover $\mathcal{V}$ of X such that $\mathcal{V} <^{\Delta} \mathcal{U}$; thus $\Delta_k(X)$ is well defined.

Corollary. *Let X be an infinite normal space. Then*

$$\dim X + 1 = \lim_{k\to\infty} \frac{\log \Delta_k(X)}{\log k}.$$

The actual derivation of this formula from the Main Theorem is not included in their paper, and so we will sketch the missing details, which are quite interesting and give a better appreciation of the Main Theorem. Let $\dim X = n$ and let $k \geqslant n+1$; in this case the formula for $\Delta_k(X)$ is a partial sum of binomial coefficients with the first term missing, and there is no closed formula for such sums. However, there are upper and lower bounds, namely

$$\binom{k}{n+1} \leqslant \Delta_k(X) \leqslant (k+1)^{n+1}$$

and we have calculations as follows:

$$\begin{aligned}\lim_{k\to\infty} \frac{\log\binom{k}{n+1}}{\log k} &= \lim_{k\to\infty} \frac{\log[k \times (k-1) \times \cdots \times (k-n)/(n+1)!]}{\log k} \\ &= \lim_{k\to\infty} \frac{\log k + \log(k-1) + \cdots + \log(k-n) - \log(n+1)!}{\log k} \\ &= n+1\end{aligned}$$

and

$$\lim_{k\to\infty} \frac{\log(k+1)^{n+1}}{\log k} = \lim_{k\to\infty} \frac{(n+1)\log(k+1)}{\log k} = n+1.$$

6.4. Cardinal functions and dimension

In his survey paper [25], Nagata points out that the following result plays a key role in the Main Theorem above: If X is a normal space with $\dim X \geqslant n \geqslant 1$, then there is a pairwise disjoint collection $\{F_k\colon\ k \geqslant 1\}$ of closed sets in X with $\dim F_k \geqslant n$ for all $k \geqslant 1$. He then asks: if $\dim X \geqslant n$, how many pairwise disjoint closed sets of $\dim \geqslant n$ can exist in X? Nagata introduces several new cardinal functions in connection with this problem. We discuss two of these; for further details, see [25]. For an infinite space X and $n \geqslant 0$ let

$$c_n(X) = \sup\{\kappa\colon \text{there exist } \kappa \text{ pairwise disjoint closed sets in } X, \text{ each of } \dim \geqslant n\};$$

$$k_n(X) = \{x\colon\ x \in X \text{ and } \dim V \geqslant n \text{ for every open neighborhood } V \text{ of } x\}.$$

Note that for $n = 0$, $c_n(X) = k_n(X) = |X|$.

Theorem 6.4. *Let X be a paracompact Hausdorff space. Then $c_n(X) \leqslant |k_n(X)|$.*

Proof. Nagata gives a sketch of the main idea; we give a few more details. Let $|k_n(X)| = \kappa$ and suppose by way of contradiction that $\kappa < c_n(X)$. Then there is a pairwise disjoint collection $\{F_\alpha\colon\ \alpha < \kappa^+\}$ of closed sets in X with $\dim F_\alpha \geqslant n$ for all $\alpha < \kappa^+$. We now obtain a contradiction by showing that for each $\alpha < \kappa^+$, there exists $x \in F_\alpha \cap k_n(X)$. If not, then for each $x \in F_\alpha$ there is an open neighborhood V_x of x such that $\dim V_x \leqslant n-1$. Now use the paracompactness of F_α to obtain a locally finite closed cover of F_α, each element of which has $\dim \leqslant n-1$ (closed subset theorem used here). By the locally finite sum theorem for closed sets, $\dim F_\alpha \leqslant n-1$, a contradiction. □

Theorem 6.5. *Let X be a normal space with* $\dim X > n$. *Then $c_n(X) \geqslant 2^\omega$.*

Outline of proof. Nagata proves that there exist disjoint closed sets F and G such that if U is any open set with $F \subseteq U \subseteq (X - G)$, then $\dim(\overline{U} - U) \geqslant n$. The proof is completed as follows. Let f be a continuous function from X into $[0,1]$ such that $f(F) = 0$ and $f(X - G) = 1$. For each t with $0 < t < 1$ let $U_t = f^{-1}([0,t))$; then $\{\overline{U}_t - U_t\colon\ 0 < t < 1\}$ is the required collection of pairwise disjoint closed sets. □

Corollary. *Let X be a paracompact Hausdorff space with* $\dim X > n$. *Then $|k_n(X)| \geqslant 2^\omega$.*

6.5. Concluding remarks

Nagata's research in dimension theory is a major achievement; for another view of the significance of this work, see [53]. Nagata has written a very interesting survey paper [31] in which he discusses open problems related to his many research interests; in the dimension theory portion he emphasizes new results and open questions related to $\triangle_k(X)$ and special metrics. Finally, we can say that the extension of dimension theory from separable metric to metric spaces owes a considerable debt to three Japanese mathematicians: K. Morita, J. Nagata, and K. Nagami.

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